Multicollisions in the Linearized Boltzmann-Landau Transport Equation

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The treatment of the linearized Boltzmann-Landau equation in the binary collision approximation given in an earlier paper [4] has been extended, under certain assumption, to include all collision processes. The effect of ternary collisions on the transport coefficients in particular has been obtained in a subsequent paper.

1. Introduction

It is believed by some that the Boltzmann-Landau equation can be applied to account for the density dependence of the transport coefficients (TC's) of a moderately dense gas through the higher functional dependence of the energy density on the distribution function, which takes account of the multicollision processes. The subject was mainly initiated by GROSSMANN [1−3] where the calculations were carried out in the lowest order Chapman-Enskog approimation in binary collisions [3]. This method was later extended by the authors [4, 5] to the first-order Chapman-Enskog approximation (i. e., by linearizing the Boltzmann-Landau equation). The calculations were however restricted to the binary collison processes only, whereas the linearization of the Boltzmann-Landau equation has been given with full generality taking account of the multicollision processes [4]. A derivation of the Vlasov form of the Boltzmann-Landau equation for a classical system has also been given using the functional Ansatz by one author [6] and in the quantum case by many authors [7, 8]. The same functional Ansatz and some other assumptions will also be used in this present work to take account of the multicollision processes allowing one to calculate the higher density corrections to the TC's beyond the contributions from the binary collisions only. The statistical system has been asumed, as before, to consist of identical particles obeying Boltzmann Statistics, i. e., a system far above the temperature of degeneracy. The notations used here will be the same as in Ref. [4] and [5].

The effect of multiple collisions is described by the multi-order functional dependence of the local energy density $e(r,t,f)$, where $f(r,t,p)$ is the distribution function. One of the basic assumptions employed here is that $e(r,t,f)$ has a functional Taylor expansion of the form (at least a truncated Taylor expansion with a desirably small remainder term)

$$e(r,t,f)=\sum_{\nu=1}^{\infty}\frac{1}{\nu!}\int F_\nu(P_1,\dots,P_\nu)\prod_{i=1}^{\nu}f(P_i)\,\mathrm{d}P_i\,. \tag{1.1}$$

The general theory of such functional Taylor expansions can be developed from nonlinear functional analysis and will be discussed elsewhere.)

Here each $F_\nu(P_1,\dots,P_\nu)$ is the ν-th functional derivative of $e(r,t,f)$ and can be taken to be symmetric in the interchange of the ν arguments $P_1,\dots,P_\nu$. [Here $F_1(P_1)=P_1^2/2m$.] The symmetry of $F_\nu(P_1,\dots,P_\nu)$ can easily be derived by an argument similar to that given earlier in ref. [6] in connection with the derivation of the Vlasov form of the Boltzmann-Landau equation.

$F_2(P_1,P_2)$, $F_3(P_1,P_2,P_3),\dots$ as is evident, are the contributions from binary, triple, ... collision processes. Using notations and explanations as in Ref. [4], the quasi-particle energy $\varepsilon_p(r,t,f)$ is given

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according to Landau theory, by

$$\delta e(r,t,f) = \int \varepsilon_{p_1}(f)\, \delta f(P_1)\, dP_1$$

where $$\varepsilon_{p_1}(f_0) = \frac{P_1^2}{2\,m} + \int \hat{F}(P_1, P_2, f)\, f(P_2)\, dP_2 \tag{1.2}$$

with $$F(P_1, P_2; f) = F_2(P_1, P_2) + \sum_{\nu=3}^{\infty} \frac{1}{(\nu-1)!} \int F_\nu(P_1, \dots, P_\nu) \prod_{i=3}^{\nu} f(P_i)\, dP_i. \tag{1.3}$$

In our previous works [4, 5] we considered only the effect of binary collision process, i. e., neglecting F_ν, $\nu > 2$. The reason was that the previous calculations were based on the assumptions that $F_2(P_1, P_2)$ was of the form $F_2(|P_1 - P_2|)$, which may not be true for $F(P_1, P_2; f)$, since higher collisions are involved. The aim of this paper is to extend the previous calculations to show the effect of higher collisions (in particular the ternary collision process) to TC's of a Boltzmann-Landau gas, under the assumption that

$$F_\nu(P_1, \dots, P_\nu) = F_\nu(\dots, |P_i - P_j|, \dots) \tag{1.4}$$

for all ν, where all pairs of indices $1 \leqq i,\ j \leqq \nu$, $i \neq j$ are considered. For example, one can consider a model in which

$$F_2(P_1, P_2) = a^2/(\mu^2 + |P_1 - P_2|^2),$$

and

$$F_3(P_1, P_2, P_3) = (F_2(P_1, P_2)\, F_2(P_2, P_3) F_2(P_3, P_1)) \cdot \left[1 + \frac{b^2}{\gamma^2 + |P_1 - P_2|^2 + |P_2 - P_3|^2 + |P_3 - P_1|^2}\right] \tag{1.6}$$

(Here a, b, μ, γ are constants.)

If we consider the effect up to ternary collisions in this model, it is easy to see that $F(P_1, P_2; f)$ does not become a function of $|P_1 - P_2|$ alone, but remains symmetric under the interchange of P_1 and P_2. The assumption (1.4) is quite plausible and makes the calculations simpler, as it can be carried out under the same arguments as employed in ref. [4, 5]. This assumption also neglects the anisotropy of the quasi-particle energy ε_p in the momentum space as well as the dependence of TC's such as the coefficients of viscosity (η) and thermal conductivity (λ) on the gradient of density n, which is the same as to neglect the effect of diffusivity on η and λ that might arise due to the anisotropy in this theory. Based on this assumption we shall discuss the linearized Boltzmann-Landau equation with full generality and calculate thereby $D f_0$ (see ref. [4]) from the conservation theorems. We shall find here and in a following paper that the non-symmetric kernels used in ref. [4] and [5] in the integral equations for the transport phenomena can be avoided and therefore we have a much simpler situation when working with a symmetric kernel. The transport coefficients have been calculated in a subsequent paper to explore the contribution, in particular, of the ternary collision process (the effect of higher collision processes can be treated similarly) to the first order density corrections to λ and η. This contribution arises mainly due to the functional dependence on f of the partial differential cross-section $\hat{\sigma}(P_1 P_2 | P_3 P_4; f)$:

$$\hat{\sigma}(P_1 P_2 | P_3 P_4; f) = \hat{\sigma}_0(P_1 P_2 | P_3 P_4) + \int \hat{\sigma}_1(P_1 P_2 | P_3 P_4; K)\, f(K)\, dK + \dots . \tag{1.7}$$

As we shall see later, $\hat{\sigma}_1$ contributes to the first order density corrections to η and λ, in a similar form obtained by ERNST et al. [9]. However the difficulty remains in computing $\hat{\sigma}_1$, even in the model constructing F_2 and F_3 according to (1.5) and (1.6). In this paper the calculations are based on the results of GROSSMANN [2] and the author's paper [4]. We shall refer to these works to avoid frequent, unnecessary repetitions.

2. The Linearized Equation, Subsidiary Conditions and Conservation Theorems

The transport phenomena in a Boltzmann-Landau gas is governed by the equation (given by Landau called Boltzmann-Landau equation by GROSSMANN; see ref. [4]).

$$D f = \frac{\partial f}{\partial t} + \frac{\partial \varepsilon_P(f)}{\partial P} \cdot \frac{\partial f}{\partial r} - \frac{\partial \varepsilon_P(f)}{\partial r} \cdot \frac{\partial f}{\partial P} = L(f) \tag{2.1a}$$

with $$L(f) = \int \sigma(P P_1 | P' P_1'; f)\, (f' f_1' - f f_1)\, dP_1\, dP_1'\, dP', \tag{2.1b}$$

$$\sigma(P P_1 | P' P_1'; f) = \delta(P + P_1 - P' - P_1')\, \delta(\varepsilon_P + \varepsilon_{P_1} - \varepsilon_{P'} - \varepsilon_{P_1'})\, \hat{\sigma}(f) \tag{2.1c}$$

where $\hat{\sigma}(f)$ is the partial differential cross-section, as defined by (1.7).

[9] M. H. ERNST, J. R. DORFMAN, and E. G. D. COHEN, Physica **31**, 493 [1965].

The linearized form of this equation is [4]

$$L(X) = \int \sigma(f_0)\, f_0(P)\, f_0(P_1) + X(P') - X(P_1) - X(P)]\, dP_1'\, dP_1\, dP' = \mathcal{D} f_0\,, \tag{2.2a}$$

where $f = f_0 + f_1$, $(f_1 \ll f_0)$, $f_1 = f_0 \cdot \varphi$, f_0 is the local equilibrium dstribution and $X(P)$ is related to φ by

$$X(P) = \varphi(P) + \beta \int \widetilde{F}(P, Q; f_0)\, f_1(Q)\, dQ \tag{2.2b}$$

with

$$\widetilde{F}(P_1, P_2; f_0) = \frac{\delta \varepsilon_{P_1}(f_0)}{\delta f(P_2)} = \frac{\delta^2 e(r,t;f_0)}{\delta f(P_1)\, \delta f(P_2)} = F_2(P_1, P_2) + \sum_{\nu=3}^{\infty} \frac{1}{(\nu-2)!} \int F_\nu(P_1, \ldots, P_\nu) \prod_{i=3}^{\nu} f(P_i)\, dP_i\,. \tag{2.2c}$$

Clearly $F(P_1, P_2; f_0)$ is symmetric in P_1 and P_2. Further $f_0(P) = \exp[\beta\{\alpha + P \cdot u - \varepsilon_P(f_0)\}]$ (2.2d)

where α, β, u have the same meaning as in ref. [4].

The Eq. (2.2a) is solvable if and only if

$$\int \psi_i(P)\, \mathcal{D} f_0\, dP = 0\,, \quad (1 \leqq i \leqq 5) \tag{2.3a}$$

where $\psi_i(P)$ are five collision invariants $1, P, \varepsilon_P(f_0)$.

As we have seen in Ref. [4], (2.3a) are just the conservation theorems subject to the subsidiary conditions

$$\int \psi_i(P)\, f_1(P)\, dP = 0 \tag{2.3b}$$

[which assures that the physical quantities $n(r,t)$, $u(r,t)$ and $e(r,t)$ are completely determined by the local equilibrium distribution f_0; see ref. [2–4]]. We shall first write the subsidiary conditions (2.3b) in terms of $X(P)$, as this will be used to obtain the transport equations in a following paper, in the form of integral equations with symmetric kernels. With definition $f_1 = f_0\, \varphi$ we write (2.2b) in the form

$$X(P) = R\, \varphi\,, \quad R = 1 + \beta\, \hat{R} \tag{2.4a}$$

where R is the integral operator defined by

$$\hat{R}\, h(P) = \int \widetilde{F}(P, Q; f_0)\, f_0(Q)\, h(Q)\, dQ\,. \tag{2.4b}$$

Assuming that R is invertible, we can write, for the same reason as given in sect. 2, Ref. [5] [compare (2.6), Ref. [5]],

$$\varphi(P) = R^{-1} X(P) = X(P) + \sum_{s=1}^{\infty} (-\beta)^s \int \widetilde{F}(P, Q_1; f_0) \ldots \widetilde{F}(Q_{s-1}, Q_s; f_0)\, X(Q_s) \sum_{i=1}^{s} f_0(Q_i)\, dQ_i\,. \tag{2.5}$$

We now assert that

$$\int f_0(P)\, \psi(P)\, (R^{-1} X)\, dP = \int f_0(P)\, X(P)\, (R^{-1} \psi)\, dP\,. \tag{2.6}$$

The proof of (2.6) runs as follows: Substituting $R^{-1} X$ from (2.5) on the left hand side of (2.6) we have

$$A \equiv \int f_0(P)\, \psi(P)\, (R^{-1} X)\, dP = \int f_0(P)\, \psi(P)\, X(P)\, dP$$
$$+ \sum_s (-\beta)^s \int \widetilde{F}(P, Q_1; f_0) \ldots \widetilde{F}(Q_{s-1}, Q_s; f_0)\, X(Q_s) \psi(P)\, f_0(P) \sum_{i=1}^{s} (f_0(Q_i)\, dQ_i)\, dP\,. \tag{2.6a}$$

Interchanging P and Q_s under the integral sign in the 2nd term of (2.6a) and then interchanging Q_1 and Q_{s-1} we have, in two steps

$$A = \int f_0(P)\, \psi(P)\, X(P)\, dP + \sum (-\beta)^s \int \widetilde{F}(Q_s, Q_1; f_0) \ldots \psi(Q_s)\, f_0(P) F(Q_{s-1}, P; f_0)\, X(P) \prod_{i=1}^{s} (f_0(Q_i)\, dQ_i)\, dP$$
$$= \int f_0(P)\, \psi(P)\, X(P)\, dP + \sum (-\beta)^s \int \widetilde{F}(Q_s, Q_{s-1}; f_0) \ldots \widetilde{F}(Q_1, P; f_0)\, X(P) \psi(Q_s)\, f_0(P) \prod_{i=1}^{s} (f_0(Q_i)\, dQ_i)\, dP$$
$$= \int f_0(P)\, X(P)\, dP\, \{\psi(P) + \sum (-\beta)^s \int \widetilde{F}(P, Q_1; f_0) \ldots \widetilde{F}(Q_{s-1}, Q_s; f_0)\, \psi(Q_s) \prod_{i=1}^{s} (f_0(Q_i)\, dQ_i) \tag{2.6b}$$

where $\widetilde{F}(P, Q; f_0) = \widetilde{F}(Q, P; f_0)$. The quantity in the parenthesis under the integral sign is just $R^{-1} \varphi$ according to (2.5), and thus we arrive at the result (2.6). The subsidiary condition, (2.3b), with $f_1 = f_0\, \varphi = f_0\, R^{-1} X$, can be put, with the help of (2.6), in the form:

$$\int f_0(P)\, X(P)\, \varphi_i(P)\, dP = 0\,, \tag{2.7a}$$

where $\varphi_i(P) = R^{-1} \psi_i(P)$, (2.7b)

and $\psi_i(P)$ being the five collision invariants. (2.7a) are the corresponding subsidiary conditions to be satisfied by $X(P)$, which in turn satisfies the linear integral equation (2.2a). Also $\varphi_i(P)$ $(1 \leqq i \leqq 5)$ are linearly independent, since R is nonsingular.

Next, the general hydrodynamic equation for the multicollision processes will be derived from the conservation theorems (2.3a). The continuity equation is easily seen to be [1, 2, 4]

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\,u) = 0 \qquad (2.8a)$$

where the mean velocity u is given by

$$u = \langle P/m \rangle_{f_0} = \langle \partial\varepsilon/\partial P \rangle_{f_0}. \qquad (2.8b)$$

Here and in the following the functional dependence of ε on f_0 is understood. For $\psi_i = P_i$, the momentum conservation theorem takes the form [1, 2, 4]

$$\frac{\partial}{\partial t}(m\,n\,u_\mu) + \frac{\partial}{\partial x_\nu} \int P_\nu \frac{\partial \varepsilon_P}{\partial P_\nu} f_0(P)\,\mathrm{d}P + \delta_{\mu\nu} \int f_0(P) \frac{\partial \varepsilon_P}{\partial x_\nu}\,\mathrm{d}P = 0. \qquad (2.9)$$

By (2.2d) we have

$$f_0(P) \frac{\partial \varepsilon_P}{\partial P_\nu} = f_0(P)\,u_\nu - k\,T \frac{\partial f_0}{\partial P_\nu}, \qquad (2.9a)$$

so that $\int P_\mu \frac{\partial \varepsilon_P}{\partial P_\nu} f_0(P)\,\mathrm{d}P = m\,n\,u_\nu\,u_\mu + n\,k\,T\,\delta_{\mu\nu}$.

Further using the symmetry property of

$$F_\nu(P_1, \ldots, P_\nu)$$

and (1.2) and (1.3), we have

$$\int f_0(P) \frac{\partial \varepsilon_P}{\partial x_\nu}\,\mathrm{d}P = \sum_{\nu=2}^{\infty} \frac{\nu-1}{\nu!} \frac{\partial}{\partial x_\nu} (n^\nu \langle F_\nu \rangle) \qquad (2.9b)$$

where $\langle A \rangle$ will mean the average of $A(P_1, \ldots, P_\nu)$ in all the variables $P_1, \ldots, P_\nu$ with respect to f_0.

(2.9a) then reads,

$$\frac{\partial}{\partial t}(m\,n\,u_\mu) + \frac{\partial}{\partial x_\nu}(m\,n\,u_\mu\,u_\nu) + \frac{\partial \pi}{\partial x_\mu} = 0 \qquad (2.10)$$

where the equilibrium scalar pressure π is given by:

$$\pi = n\,k\,T + \frac{n^2}{2}\langle F_2 \rangle + \frac{n^3}{3!}\langle F_3 \rangle + \ldots + \frac{\nu-1}{\nu!} n^\nu \langle F_\nu \rangle + \ldots. \qquad (2.11)$$

The type of virial expansion (2.11) has also been derived by GROSSMANN [2].

Finally, the energy conservation equation is given by [2–4] to be

$$\frac{\partial}{\partial t}(n\langle\varepsilon\rangle) - \int f_0(P) \frac{\partial \varepsilon_P}{\partial t}\,\mathrm{d}P + \operatorname{div} Q = 0 \qquad (2.12)$$

where the heat flow Q is defined by

$$Q_\nu = \int \varepsilon_P \frac{\partial \varepsilon_P}{\partial P_\nu} f_0(P)\,\mathrm{d}P = n\,u_\nu(\langle\varepsilon\rangle + k\,T). \qquad (2.13)$$

The right hand side of (2.13) can be easily verified using (2.9a). Also, from definitions (1.1) and (1.2) it readily follows that

$$n\langle\varepsilon\rangle = e(r,t) + \sum_{\nu=2}^{\infty}\left[\frac{1}{(\nu-1)!} - \frac{1}{\nu!}\right] \cdot \int F_\nu(P_1, \ldots, P_\nu) \prod_{\nu=2}^{\infty} (f(P_i)\,\mathrm{d}P_i) \qquad (2.14)$$
$$= e(r,t) + \pi - n\,k\,T,$$

so that $$Q_\nu = u_\nu[e(r,t) + \pi]. \qquad (2.15)$$

In the same way as we arrived at the Eq. (2.9b) we can deduce that

$$\int f_0(P) \frac{\partial \varepsilon_P}{\partial t}\,\mathrm{d}P = \sum_{\nu=2}^{\infty} \frac{(\nu-1)}{\nu!} \frac{\partial}{\partial t}(n^\nu \langle F_\nu \rangle). \qquad (2.16)$$

(2.12) then takes the form

$$\frac{\partial e(r,t)}{\partial t} + \operatorname{div} Q = 0. \qquad (2.17)$$

We shall now use the conservation equations in the form (2.8), (2.10) and (2.17) to calculate $\mathcal{D} f_0$ and complete the aim of this paper.

3. Calculation of $\mathcal{D} f_0$

According to Ref. [4] we have

$$\mathcal{D} f_0 = \frac{\partial f_0}{\partial n} \cdot D\,n + \frac{\partial f_0}{\partial u_i} D\,u_i + \frac{\partial f_0}{\partial T} D\,T - \frac{\partial \varepsilon_P}{\partial x_i} \cdot \frac{\partial f_0}{\partial P_i} \qquad (3.1)$$

where $$D = \frac{\partial}{\partial t} + \frac{\partial \varepsilon_P}{\partial P_i} \frac{\partial}{\partial x_i} \qquad (3.2)$$

and ε_P depends functionally on f_0. We now make use of our assumption (1.4). This assumption makes

$$F_\nu(P_1) = \frac{\int F_\nu(P_1, \ldots, P_\nu) \prod_{i=2}^{\nu} (f(P_i)\,\mathrm{d}P_i)}{\int \prod_{i=2}^{\nu} (f(P_i)\,\mathrm{d}P_i)} \qquad (3.3)$$

and therefore $\varepsilon_{P_1}(f_0)$ a function of $p_1 = |P_1 - m\,u|$ alone, where $p_1 = P_1 - m\,u$. Thus the energy $\varepsilon_P(f_0)$ is isotropic in the momentum space of p_s. To prove this statement we use the symmetry property of $F_\nu(P_1, \ldots, P_\nu)$ and the virial expansion

$$F_\nu(P_1) = [F_\nu(P_1)]_{n=0} + n\left[\frac{\mathrm{d}F_\nu(P_1)}{\mathrm{d}n}\right]_{n=0} + \ldots \qquad (3.4)$$

Now

$$[F_\nu(P_1)]_{n=0} \tag{3.5}$$
$$= \int F_\nu(\ldots, |P_1 - P_j|, \ldots) \prod_{i=1}^{\nu} (w_0(P_i)\, dP_i),$$

where

$$w_0(P) = \frac{\exp[-\beta\, p^2/2\, m]}{\int \exp[-\beta\, p^2/2\, m]\, dp} = \left[\frac{f_0(P)}{\int f_0(P)\, dp}\right]_{n=0} \tag{3.6}$$

is the normalized Maxwell distribution.

If we integrate (3.5) first with respect to $P_3, \ldots, P_\nu$, the result will be, by virtue of symmetry [compare (2.2c) in relation to the symmetric function $F(P_1, P_2)$], a function $K(|P_1 - P_2|, P_1^2, P_2^2, P_1 \cdot P_2)$ which must also be symmetric in the interchange of P_1 and P_2. A second integration with respect to P_2 will yield $F_\nu(P_1)$ as a function of $p_1 = |p_1|$ alone. We however note that K need not be a function of $|P_1 - P_2|$ alone, in agreement with the remark made at the end of ref. [4]. Each term in the virial expansion (3.5) can be treated in the same way to obtain the desired result [see e. g. (2.3)].

As in Ref. [4], $f_0(P)$ can be similarly expressed as a function of p alone; i. e.,

$$f_0(P) = \frac{n}{Z(n, T)} \exp[-\beta\, \varepsilon_p(f_0)] \tag{3.7a}$$

$$\text{where} \quad Z(n, T) = \int \exp[-\beta\, \varepsilon_p(f_0)]\, dp \tag{3.7b}$$

which follows from the relation $n = \int f_0(P)\, dP$ used to calculate $\alpha\,\beta$ appearing in f_0 (see Ref. [2]).

Under this assumption we can now calculate $e(r, t)$. Using (2.9a) it is easy to verify that

$$\sum_\mu \int P \frac{\partial \varepsilon_P}{\partial P_\mu} f_0(P)\, dP = m\, n\, u^2 + 3\, n\, k\, T. \tag{3.8a}$$

On the other hand

$$\frac{\partial \varepsilon_P}{\partial P_\mu} = \frac{P_\mu}{m} + n \frac{\partial \hat{F}(p, f_0)}{\partial P_\mu}$$

where $\hat{F}(P_1, f_0) = \int \hat{F}(P_1, P_2, f_0)\, dP_2$ depends on P_1 only through its dependence on p_1.

Substituting in (3.8a) we get

$$\tfrac{1}{2}\, m\, n\, u^2 + \tfrac{3}{2}\, n\, k\, T = \int \frac{P^2}{2\, m} f_0(P)\, dP + \sum_{\mu=1}^{3} \frac{n}{2} \int P_\mu \frac{\partial \hat{F}(p, f_0)}{\partial P_\mu} f_0(P)\, dP$$
$$= \int \frac{P^2}{2\, m} f_0(P)\, dP + \frac{n}{2} \sum_{\mu=1}^{3} \int P_\mu \frac{\partial \hat{F}(p, f_0)}{\partial p_\mu} f_0(P)\, dP + \sum \frac{n\, u_\mu}{2} \int \frac{\partial \hat{F}(p, f_0)}{\partial p_\mu} f_0(p)\, dp. \tag{3.8b}$$

It is easy to see that under the restriction imposed on $F_\nu(P_1, \ldots, P_\nu)$, the 3rd term on the right hand side of (3.8b) vanishes, since

$$\int \frac{\partial F_\nu(p_1, \ldots, p_\nu)}{\partial p_{1\mu}} \prod_{i=1}^{\nu} (f(p_i)\, dp_i)$$
$$= \frac{1}{\nu} \int \left[\sum_{i=1}^{\nu} \frac{\partial F_\nu(p_1, \ldots, p_\nu)}{\partial p_{i\mu}}\right] \prod_{i=1}^{\nu} (f(p_i)\, dp_i)$$

$$\text{and} \quad \sum_{i=1}^{\nu} \frac{\partial F_\nu}{\partial p_{i\mu}} = 0. \tag{3.8b}$$

Using (3.8b) it is not difficult to verify (2.8b) in the same way as has been done by Grossmann in the binary process. Let now

$$\hat{\zeta}(n, T) = \sum_{\mu=1}^{3} \int p_\mu \frac{\partial \hat{F}(p, f_0)}{\partial p_\mu} f_0(p)\, dp$$
$$= \int p \frac{\partial \hat{F}(p, f_0)}{\partial p} f_0(p)\, dp \tag{3.9}$$

which is independent of u. $\hat{\zeta}(n, T)$ is the general form of $\zeta(n, T)$ of Eq. (4.6b) of Ref. [4], taking into account the effect of multicollisions through the functional dependence of $\hat{F}$ on f_0.

Then from (1.1) and (3.8b) we have:

$$e(r, t) = \frac{m\, n\, u^2}{2} + \frac{3}{2}\, n\, k\, T - \frac{n}{2} \hat{\zeta}(n, T) - \sum_{\nu=2}^{\infty} \frac{n^\nu}{\nu!} \langle F_\nu \rangle. \tag{3.10}$$

Since $\langle F_\nu \rangle$ does not depend explicitly on u, we have the identity

$$\frac{1}{m\, n} \frac{\partial e}{\partial u_\mu} - u_\mu = 0. \tag{3.11}$$

Together with (4.11) we shall need the identities:

$$2\left\langle \frac{\partial \hat{F}(p_1, f_0)}{\partial n} \right\rangle - \frac{\partial \langle \hat{F}(f_0) \rangle}{\partial n} - \left\langle \frac{\partial \hat{F}(p_1, f_0)}{\partial n} \right\rangle = 0 \tag{3.12a}$$

and

$$2\left\langle \frac{\partial \hat{F}(p_1, f_0)}{\partial T} \right\rangle - \frac{\partial \langle \hat{F}(f_0) \rangle}{\partial T} - \left\langle \frac{\partial \hat{F}(p_1, f_0)}{\partial T} \right\rangle = 0 \tag{3.12b}$$

to find $D f_0$.

These go over to the forms (4.10) and (4.11) of Ref. [4] in the case of the binary collision approximation for which $\hat{F}(p, q; f_0) = F(p, q)$ is independent of f_0. We shall prove (3.12a), whereas (3.12b) will follow similarly. We have, by definition

$$n\left\langle\frac{\partial\hat{F}(p_1,f_0)}{\partial n}\right\rangle$$
$$=\int f_0(p_1)\frac{\partial}{\partial n}\left\{\frac{\int\hat{F}(p_1,p_2,f_0)\,f_0(p_2)\,\mathrm{d}p_2}{n}\right\}\mathrm{d}p_1$$
$$=\frac{1}{n}\int\frac{\partial\hat{F}(p_1,p_2,f_0)}{\partial n}f_0(p_1)\,f_0(p_2)\,\mathrm{d}p_1\,\mathrm{d}p_2$$
$$+\frac{1}{n}\int f_0(p_1)\,\hat{F}(P_1,P_2,f_0)\frac{\partial f_0(p_2)}{\partial n}\mathrm{d}p_1\,\mathrm{d}p_2$$
$$-\frac{1}{n^2}\int\hat{F}(p_1,p_2,f_0)\,f_0(p_1)\,f_0(p_2)\,\mathrm{d}p_1\,\mathrm{d}p_2$$
$$=\frac{1}{n}\int\frac{\partial\hat{F}(P_1,P_2,f_0)}{\partial n}f_0(p_1)\,f_0(p_2)\,\mathrm{d}p_1\,\mathrm{d}p_2$$
$$+\frac{1}{2n}\int\hat{F}(p_1,p_2,f_0)\frac{\partial}{\partial n}\{f_0(P_1)\,f_0(P_2)\}\,\mathrm{d}p_1\,\mathrm{d}p_2$$
$$-\langle\hat{F}(f_0)\rangle$$
$$=n\left\langle\frac{\partial\hat{F}(f_0)}{\partial n}\right\rangle$$
$$+\frac{1}{2n}\frac{\partial}{\partial n}\int\hat{F}(p_1,p_2,f_0)\,f_0(p_1)\,f_0(p_2)\,\mathrm{d}p_1\,\mathrm{d}p_2$$
$$-\frac{1}{2n}\int\frac{\partial F(p_1,p_2,f_0)}{\partial n}f_0(p_1)\,f_0(p_2)\,\mathrm{d}p_1\,\mathrm{d}p_2$$
$$-\langle\hat{F}(f_0)\rangle$$
$$=n\left\langle\frac{\partial\hat{F}(f_0)}{\partial n}\right\rangle+\frac{1}{2n}\frac{\partial}{\partial n}\left(n^2\langle\hat{F}(f_0)\rangle\right)$$
$$-\frac{n}{2}\left\langle\frac{\partial\hat{F}(f_0)}{\partial n}\right\rangle-\langle\hat{F}(f_0)\rangle$$
$$=\frac{n}{2}\left\langle\frac{\partial\hat{F}(f_0)}{\partial n}\right\rangle+\frac{n}{2}\frac{\partial}{\partial n}\langle\hat{F}(f_0)\rangle\,.$$

(Here we have used the symmetry of $\widetilde{F}(p_1,p_2,f_0)$ in the interchange of p_1 and p_2.) The identity (3.12a) follows directly.

The above identities are used to calculate the quantities $D\,n$, $D\,u_\mu$, $D\,T$. The results are the same as those given in Ref. [4] except that F in Ref. [4] is now replaced by $\hat{F}(f_0)$. The calculation is simple, but long. We produce the results here for further references.

$$D\,n=\frac{p_i}{m}S_1(p)\frac{\partial n}{\partial x_i}-n\,\mathrm{div}\,u \qquad (3.13a)$$

where

$$S_1(p)=\left(1+\frac{n\,m}{P}\frac{\partial\hat{F}(p,f_0)}{\partial p}\right). \qquad (3.13b)$$

$$D\,u_i=\frac{p_j}{m}\frac{\partial u_i}{\partial x_j}S_1(p)-\frac{1}{m\,n}\frac{\partial\pi}{\partial x_i}, \qquad (3.13c)$$

$$D\,T=\frac{p_i}{m}\frac{\partial T}{\partial x_i}S_i(p)+\Lambda_{ij}\,\delta_{ij}\,Y(n,u,T) \qquad (3.13d)$$

where

$$Y(n,u,T)=\left(n\frac{\partial e}{\partial n}-e-\pi\right)\Big/\frac{\partial e}{\partial T}$$

and

$$\Lambda_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right).$$

Using (3.13) we get

$$D\,f_0=f_0(P)\left[\frac{1}{T}\frac{\partial T}{\partial x_i}p_i\,R(p)+\Lambda_{ij}\,S_{ij}(p)\right] \qquad (3.14)$$

where

$$R(p)=\frac{1}{m}\left[R_0(p)+n\,\beta\,R_1(p)\right]S_1(p), \qquad (3.15)$$

$$R_0(p)=\frac{\beta\,p^2}{2\,m}-\frac{5}{2}$$
$$R_1(p)=\hat{F}(p;f_0)-\langle\hat{F}\rangle+\frac{1}{2\,n}\hat{\zeta}(n,T), \qquad (3.16)$$

$$S_{ij}=\frac{\beta}{m}\left(p_i\,p_j-\frac{p^2}{3}\delta_{ij}\right)S_1(p)+\delta_{ij}\,S_2(p), \qquad (3.17)$$

and $$S_2(p)=\frac{Y}{f_0}\frac{\partial f_0}{\partial T}-\frac{n}{f_0}\frac{\partial f_0}{\partial n}+\frac{\beta\,p^2}{3\,m}S_1(p). \qquad (3.18)$$

(2.2a) with $D\,f_0$ given by (3.14) forms the linear symmetric integral equation for the Boltzmann-Landau transport phenomena, subject to the subsidiary conditions given by (2.7a). The subsidiary conditions contain $\widetilde{F}(P,Q;f_0)$ which again depends on P and Q through $p=P-m\,u$ and $q=Q-m\,u$ only.

Though the results of this section are quite similar to the forms given in ref. [4], they are probably more general, since we have taken the multicollision processes into account (through the functional dependence of $\hat{F}$ on f_0) under the special assumption (1.4) on F_ν s. This assumption in fact leads to these relatively simplified forms of ref. [4], where grad n does not appear in $D\,f_0$, which is naturally expected from earlier works.

The more general calculations may not seem at this time to be of much interest until certain other critical situations regarding logarithmic divergences and numerical computations are well-understood. In a subsequent paper we shall calculate formally the transport coefficients λ and η (using the symmetric equation, instead of the non-symmetric one) and consider particularly the effect of ternary collisions to the first order density corrections.

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